SUCCESSIVE RADII OF FAMILIES OF CONVEX BODIES

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Abstract. We study properties of the so called in- and outer successive radii of special families of convex bodies. First we consider the balls of the p-norms, for which we show that the precise value of the outer (inner) radii when $p \geq 2$ ($1 \leq p \leq 2$), as well as bounds otherwise, can be obtained as consequences of known results on Gelfand and Kolmogorov numbers of identity operators between finite dimensional normed spaces. We also prove properties that successive radii satisfy when we restrict to the families of the constant width sets and the p-tangential bodies.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|_2$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively, and denote by e_i the i-th canonical unit vector. For a 0-symmetric convex body $K \in \mathcal{K}^n$, i.e., such that $K = -K$, the well-known Minkowski functional $\min\{\lambda \geq 0 : x \in \lambda K\}$ defines a norm, denoted by $|\cdot|_K$, which has K as its unit ball.

The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$, and for $L \in \mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement. For $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$, the orthogonal projection of K onto L is denoted by $K|L$. With $\operatorname{lin}\{u_1, \dots, u_m\}$ we represent the linear hull of the vectors $u_1, \dots, u_m$ and with $[u_1, u_2]$ the line segment with end-points u_1, u_2. Finally, for $S \subset \mathbb{R}^n$ we denote by $\operatorname{conv} S$ the convex hull of S and by $\operatorname{int} S$ and $\operatorname{bd} S$ its interior and boundary. Moreover, we write $\operatorname{relbd} S$ to represent its relative boundary, i.e., the boundary of S relative to its affine hull $\operatorname{aff} S$, and $\dim S$ to represent the dimension of the set, i.e., the dimension of $\operatorname{aff} S$.

The diameter, minimal width, circumradius and inradius of a convex body K are denoted by $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For more information on these functionals and their properties we refer to [5, pp. 56–59]. If f is a functional on $\mathcal{K}^n$ depending on the dimension in which a convex body K is embedded, and if K is contained in an affine space A, then we write $f(K; A)$ to stress that f has to be evaluated with respect to the space A. Successive outer and inner radii are defined in the following way.

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Definition 1.1. *For $K\in\mathcal{K}^n$ and $i=1,\dots,n$ let*

$$\mathrm{R}_i(K)=\min_{L\in\mathcal{L}_i^n}\mathrm{R}(K|L)\quad \text{ and }\quad \mathrm{r}_i(K)=\max_{L\in\mathcal{L}_i^n}\max_{x\in L^\perp}\mathrm{r}\bigl(K\cap(x+L);x+L\bigr).$$

It is clear that the outer radii are increasing in i, whereas the inner radii are decreasing in i. Moreover, $\mathrm{R}_i(K)$ is the smallest radius of a solid cylinder with i-dimensional spherical cross section containing K, and $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K. We obviously have $\mathrm{R}_n(K)=\mathrm{R}(K)$, $\mathrm{R}_1(K)=\omega(K)/2$, $\mathrm{r}_n(K)=\mathrm{r}(K)$ and $\mathrm{r}_1(K)=\mathrm{D}(K)/2$.

The first systematic study of these and other families of successive radii was developed in [2]. For more information on these radii and their relation with other measures, we refer, for instance, to [4, 15, 19, 20, 28] and the references inside. The successive radii of Definition 1.1 are closely related to some notions in approximation theory, namely, they are particular cases of the so-called Gelfand and Kolmogorov numbers of identity operators between finite dimensional normed spaces (see e.g. [23, 26]), which will be defined in Section 2.

Here we are interested in computing/studying properties of the radii of special families of convex bodies. So far, only orthogonal boxes, orthogonal cross-polytopes [8, 13], simplices [1, 3, 8, 9] and ellipsoids [18] have been studied and their radii explicitly given. In this paper we mainly consider two families of convex bodies:

i) Unit p-balls: for $p\geq 1$ we denote by B_p^n the unit p-ball associated to the p-norm $|\cdot|_p$, i.e.,

$$B_p^n=\left\{x=(x_1,\dots,x_n)\in\mathbb{R}^n:|x|_p=\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}\leq 1\right\},$$

with $|x|_\infty=\max\{|x_i|:i=1,\dots,n\}$. The normed space with unit ball B_p^n is as usual denoted by ℓ_p^n. For the sake of brevity, we will write $B_n=B_2^n$ to denote the n-dimensional Euclidean unit ball. Moreover, for $L\in\mathcal{L}_i^n$, we will write $B_{i,L}=B_n\cap L$. Notice that when $p=1$ and $p=\infty$ the (unit) p-balls are, respectively, the regular cross-polytope $B_1^n=\mathrm{conv}\{\pm\mathrm{e}_1,\dots,\pm\mathrm{e}_n\}$ and the regular cube $B_\infty^n=\sum_{j=1}^n[-\mathrm{e}_j,\mathrm{e}_j]$ with edge-length 2; here $+$ denotes the usual Minkowski (vectorial) addition. The values of the successive radii of these particular p-balls are (see e.g. [8])

$$\mathrm{R}_i(B_\infty^n)=\sqrt{i},\quad \mathrm{r}_i(B_\infty^n)=\sqrt{\frac{n}{i}},\quad \mathrm{R}_i(B_1^n)=\sqrt{\frac{i}{n}},\quad \mathrm{r}_i(B_1^n)=\sqrt{\frac{1}{i}}. \tag{1}$$

ii) Constant width sets: A convex body $K\in\mathcal{K}^n$ has constant width if it has the same width b in all directions, i.e., if its diameter and minimal width have the same value, $\mathrm{D}(K)=\omega(K)=\mathrm{b}$. The class of convex bodies of constant width will be denoted by $\mathcal{W}^n$. For a nice and thorough survey on convex bodies of constant width see [11].

In Section 2 we introduce the Gelfand and Kolmogorov numbers, stating the necessary notation from Banach space theory and approximation theory. We also present some known properties of these numbers and show the relation between them and the successive radii of Definition 1.1.

Section 3 is devoted to study the in- and outer radii of the (unit) p-balls, obtaining the following results.

Theorem 1.1. *Let $p \geq 2$ and $1 \leq q \leq 2$. For all $i = 1, \dots, n$ it holds*

$$\mathrm{R}_i(B_p^n) = i^{1/2-1/p} \quad \textit{and} \quad \mathrm{r}_i(B_q^n) = i^{1/2-1/q}.$$

In the case $1 \leq p \leq 2$ (respectively, $q \geq 2$) we give matching upper and lower bounds for the outer (inner) radii which are collected in the next theorem. Here and later we use the notation $a_{n,i} \asymp b_{n,i}$ for some double sequences $a_{n,i}, b_{n,i}$ of non-negative real numbers to mean that there exist absolute constants $c, C > 0$ such that $c\,a_{n,i} \leq b_{n,i} \leq C\,a_{n,i}$.

Theorem 1.2. *Let $1 \leq p \leq 2$ and $q \geq 2$. For all $i = 1, \dots, n$ it holds*

$$\mathrm{R}_i(B_p^n) \asymp \begin{cases} \left(\frac{i}{n}\right)^{1/2} & \textit{for } \ i \geq n^{2(1-1/p)}, \\ n^{1/2-1/p} & \textit{for } \ i \leq n^{2(1-1/p)}, \end{cases}$$

and

$$\mathrm{r}_i(B_q^n) \asymp \begin{cases} \left(\frac{n}{i}\right)^{1/2} & \textit{for } \ i \geq n^{2/q}, \\ n^{1/2-1/q} & \textit{for } \ i \leq n^{2/q}. \end{cases}$$

In Section 4 we study the relation between the radii and the constant width sets. It is well known (see e.g. [12, p. 125]) that if $K \in \mathcal{W}^n$ with width b, then the inball and the circumball of K are concentric and both,

$$\mathrm{R}(K) + \mathrm{r}(K) = \mathrm{b} \quad \text{and} \quad \mathrm{D}(K) + \omega(K) = 2\mathrm{b}. \tag{2}$$

So the natural question arises if an analogous relation holds for the more general in- and outer radii, namely,

$$\mathrm{R}_i(K) + \mathrm{r}_i(K) = \mathrm{b}, \quad i = 1, \dots, n. \tag{3}$$

The next theorem shows that this relation is, in general, not true except, of course, when $i = 1, n$.

Theorem 1.3. *Let $K \in \mathcal{W}^n$ with width* b. *Then $\mathrm{R}_i(K) + \mathrm{r}_i(K) \leq \mathrm{b}$, and the inequality can be strict, as the Meissner body shows.*

It can be easily seen (see Proposition 4.1) that for a different definition of inner radii in which projections are involved, it is possible to get an equality relation of the type (3). Some additional properties of the radii of constant width sets are also studied.

Finally, in Section 5 we consider an additional family of convex bodies, the so-called p-tangential bodies, for which a nice relation between their inner radii can be proved.

2. Gelfand numbers, Kolmogorov numbers and successive radii of symmetric convex bodies

The authors of [15] already mentioned the close relation of successive radii to notions of width studied in approximation theory, see e.g. [10, 26, 27]. Nevertheless, it seems that up to now this intimate connection has not been so far highlighted in its full generality. So some results proved for successive radii in recent years can be translated from corresponding results about Gelfand numbers and Kolmogorov numbers of identity operators between finite dimensional normed spaces. Our aim in this section is to point out the formal connection between successive radii and Gelfand and Kolmogorov numbers and to translate results from approximation theory to the geometric setting of successive radii.

We start by introducing the necessary notation from Banach space theory and from approximation theory. The letters X, Y always stand for Banach spaces. The dual space of all bounded linear functionals on X will be denoted by X'. In this particular setting, we will also represent the action of $a \in X'$ on $x \in X$ by $\langle x, a\rangle$. The Banach space $\mathcal{L}(X,Y)$ is the space of all linear bounded operators from X to Y with the usual operator norm, denoted by $\|\cdot\|$. Then, the dual operator $T' \in \mathcal{L}(Y', X')$ of $T \in \mathcal{L}(X,Y)$ is given by $\langle x, T'b\rangle = \langle Tx, b\rangle$ for $x \in X$ and $b \in Y'$. It satisfies $\|T'\| = \|T\|$.

For $T \in \mathcal{L}(X,Y)$, we define the *k-th approximation number* as

$$a_k(T) := \inf\bigl\{\|T - R\| : R \in \mathcal{L}(X,Y),\, \operatorname{rank} R < k\bigr\},$$

the *k-th Gelfand number* as

$$c_k(T) := \inf\left\{\|T_{|_M}\| : M \text{ linear subspace of } X,\, \operatorname{codim} M < k\right\},$$

and the *k-th Kolmogorov number* as

$$d_k(T) := \inf\bigl\{\|q_N T\| : N \text{ linear subspace of } Y,\, \dim N < k\bigr\};$$

here $T_{|_M}$ is the restriction of T to the subspace M and q_N denotes the quotient mapping $Y \longrightarrow Y/N$.

More explicit descriptions of the Gelfand and Kolmogorov numbers are

$$c_k(T) = \inf_{\substack{M \subset X \\ \operatorname{codim} M < k}} \ \sup_{x \in M, \|x\| \le 1} \|Tx\|$$

and

$$d_k(T) = \inf_{\substack{N \subset Y \\ \dim N < k}} \ \sup_{x \in X, \|x\| \le 1} \ \inf_{y \in N} \|Tx - y\|.$$

In the following lemma we collect some basic known facts about these quantities. For this and more information on s-numbers of operators in the normed case we refer to [23, 26].

Lemma 2.1. *Let $s \in \{a, c, d\}$, $k \in \{1, \dots, n\}$ and $T \in \mathcal{L}(X,Y)$. Then:*

i) $\|T\| \ge s_1(T) \ge s_2(T) \ge s_3(T) \ge \cdots \ge 0$.

ii) $s_k(STR) \le \|S\| s_k(T) \|R\|$, *for all operators R, S for which the product STR is defined.*

iii) $c_k(T) \leq a_k(T)$ *and* $d_k(T) \leq a_k(T)$.
iv) $c_k(T) = a_k(T)$ *whenever* X *is a Hilbert space and* $d_k(T) = a_k(T)$ *whenever* Y *is a Hilbert space.*
v) $a_k(T') = a_k(T)$, *and* $d_k(T') = c_k(T)$ *whenever* T *is a compact operator between Banach spaces.*

In order to state the connection of the above numbers with the successive radii, we need the well-known correspondence between a 0-symmetric convex body $K \in \mathcal{K}^n$ and the n-dimensional normed space $X_K = \big(\mathbb{R}^n, |\cdot|_K\big)$ with unit ball K. For two such bodies K and E, let I_K^E denote the identity operator of $\mathbb{R}^n$ considered as an operator between the corresponding normed spaces, $X_K \longrightarrow X_E$. If $K = B_p^n$, then we abbreviate I_p^E for I_K^E. Similarly, if $E = B_q^n$, we write I_K^q for I_K^E. Now the notation I_p^q is self-explaining.

Let $K^* = \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \text{ for all } x \in K\big\}$ denote, as usual, the polar body of K. Recall that K^* is the unit ball of the dual space of X_K, i.e., $X_K' = X_{K^*}$. Moreover, it holds

$$\big(I_K^E\big)' = I_{E^*}^{K^*}. \tag{4}$$

The following theorem gives the formal connection between the Gelfand and Kolmogorov numbers, and the successive radii.

Theorem 2.1. *Let* $K \in \mathcal{K}^n$ *be* 0*-symmetric. For all* $i = 1, \ldots, n$ *it holds*

$$\mathrm{r}_i(K) = c_{n-i+1}\big(I_2^K\big)^{-1} = d_{n-i+1}\big(I_{K^*}^2\big)^{-1} = a_{n-i+1}\big(I_2^K\big)^{-1}$$

and

$$\mathrm{R}_i(K) = d_{n-i+1}\big(I_K^2\big) = c_{n-i+1}\big(I_2^{K^*}\big) = a_{n-i+1}\big(I_K^2\big).$$

Proof. The last two equalities between the Gelfand, Kolmogorov and approximation numbers follow immediately from the properties of these numbers stated above (see Lemma 2.1 and (4)).

For a 0-symmetric convex body K, the definition of $\mathrm{r}_i(K)$ reduces to

$$\mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K \cap L; L).$$

Let $L \in \mathcal{L}_i^n$ be any i-dimensional linear subspace of $\mathbb{R}^n$. Observe that

$$\big\|I_2^K{}_{|L}\big\| = \min\big\{R > 0 : |x|_K \leq R|x|_2 \text{ for all } x \in L\big\}$$

and

$$\begin{aligned}\mathrm{r}(K \cap L; L) &= \max\big\{r > 0 : rB_{i,L} \subset K \cap L\big\}\\ &= \max\left\{r > 0 : |x|_K \leq \frac{1}{r}|x|_2 \text{ for all } x \in L\right\}.\end{aligned}$$

Thus it follows that

$$\mathrm{r}(K \cap L; L) = \big\|I_2^K{}_{|L}\big\|^{-1},$$

and taking the maximum over $L \in \mathcal{L}_i^n$, which is the same as taking the maximum over all L with $\operatorname{codim} L < n-i+1$, we get $\mathrm{r}_i(K) = c_{n-i+1}\big(I_2^K\big)^{-1}$.

The equality for the outer radii is now deduced from the duality relation $d_{n-i+1}\big(I_K^2\big) = c_{n-i+1}\big(I_2^{K^*}\big)$ stated above, the previously proved identity $\mathrm{r}_i(K^*) = c_{n-i+1}\big(I_2^{K^*}\big)^{-1}$, and the known relation (see [15, (1.2)])

$$\mathrm{R}_i(K)\,\mathrm{r}_i(K^*) = 1. \tag{5}$$

We would like to emphasize that (5) can be also seen as a special case of the duality relation $c_k(T') = d_k(T)$ between Gelfand and Kolmogorov numbers. For completeness we give a self-contained short argument.

To this end, observe that

$$\begin{aligned}\mathrm{R}(K|L) &= \min\{R>0 : K|L \subset RB_n\}\\ &= \min\big\{R>0 : |P_L x|_2 \le R|x|_K \text{ for all } x\in\mathbb{R}^n\big\}\\ &= \big\|P_L I_K^2\big\|,\end{aligned}$$

where P_L denotes the orthogonal projection onto L in the Euclidean space ℓ_2^n. Then it follows from [25, Proposition 11.6.2] that

$$\begin{aligned}\mathrm{R}_i(K) &= \min_{L\in\mathcal{L}_i^n}\mathrm{R}(K|L) = \min_{L\in\mathcal{L}_i^n}\big\|P_L I_K^2\big\| = d_{n-i+1}\big(I_K^2\big)\\ &= c_{n-i+1}\big(I_2^{K^*}\big) = \frac{1}{\mathrm{r}_i(K^*)}.\end{aligned}$$

□

3. Successive radii of p-balls

In this section we use the general characterization of inner and outer successive radii by approximation quantities given in Theorem 2.1 to deduce exact values and sharp asymptotic estimates for successive radii of p-balls. This also shows that the results for $p=1$ and $p=\infty$ referred to in (1) can be also derived from known results about Gelfand and Kolmogorov numbers.

We start collecting the known results for Gelfand and Kolmogorov numbers $c_k(I_2^p)$ and $d_k(I_p^2)$ for $1\le p\le\infty$. It was proved by Steckin [31] and Pietsch [24] that for all $k=1,\dots,n$,

$$d_k(I_1^2) = c_k(I_2^\infty) = \sqrt{\frac{n-k+1}{n}} \quad \text{and} \quad c_k(I_1^2) = d_k(I_2^\infty) = \sqrt{n-k+1}.$$

By Theorem 2.1 this immediately implies (1). Pietsch actually computed all s-numbers

$$a_k(I_p^q) = c_k(I_p^q) = d_k(I_p^q) = (n-k+1)^{1/q-1/p}$$

when $1\le q\le p\le\infty$. In particular, it holds

$$d_k(I_p^2) = (n-k+1)^{1/2-1/p} \quad \text{and} \quad c_k(I_2^q) = (n-k+1)^{1/q-1/2}$$

for $2\le p\le\infty$ and $1\le q\le 2$. Then, using Theorem 2.1, we get as a direct consequence for successive radii Theorem 1.1.

The computation of the remaining Kolmogorov and Gelfand numbers of identity operators I_p^q turned out to be more complicated. In the relevant cases for us, the exact values seem to be very difficult to determine. Nevertheless, matching lower and upper bounds up to multiplicative constants are known. The result we need is due to Gluskin [16], who proved that, for $q \geq 2$,

$$c_k(I_2^q) \asymp \begin{cases} \left(\frac{n-k+1}{n}\right)^{1/2} & \text{for } 1 \leq k \leq n+1-n^{2/q}, \\ n^{1/q-1/2} & \text{for } n+1-n^{2/q} \leq k \leq n, \end{cases}$$

and, by duality, for $1 < p \leq 2$,

$$d_k(I_p^2) \asymp \begin{cases} \left(\frac{n-k+1}{n}\right)^{1/2} & \text{for } 1 \leq k \leq n+1-n^{2(1-1/p)}, \\ n^{1/2-1/p} & \text{for } n+1-n^{2(1-1/p)} \leq k \leq n. \end{cases}$$

By Theorem 2.1, the direct consequence for successive radii is Theorem 1.2.

In connection with the approximation of embeddings between function spaces, considerable work has been done to compute the Gelfand and Kolmogorov numbers of diagonal operators. We will now translate some of this work into results for successive radii. Let D_t be the diagonal matrix with diagonal $t = (t_1, \ldots, t_n)$, considered as a map on $\mathbb{R}^n$. We will always assume that $t_1 \geq t_2 \geq \cdots \geq t_n > 0$. The following result is a special case of [25, Theorem 11.11.4].

Proposition 3.1. *Let $1 \leq q \leq 2$ and $p \geq 2$ and define positive numbers r, s by $1/r = 1/q - 1/2$ and $1/s = 1/2 - 1/p$. Then*

$$c_k\bigl(D_t : \ell_2^n \longrightarrow \ell_q^n\bigr) = \left(\sum_{j=k}^n t_j^r\right)^{1/r} \quad \textit{and} \quad d_k\bigl(D_t : \ell_p^n \longrightarrow \ell_2^n\bigr) = \left(\sum_{j=k}^n t_j^s\right)^{1/s}.$$

Let $K_p = D_t(B_p^n)$, $p \geq 2$, and $K^q = D_t^{-1}(B_q^n)$, $1 \leq q \leq 2$. This is, K_p and K^q are orthogonally dilated images of the balls B_p^n and B_q^n, respectively, t_i and t_i^{-1} being the respective lengths of the half-axes in the direction e_i. Thus from the properties of the Gelfand and Kolmogorov numbers, we directly obtain from Proposition 3.1 that

$$c_k\bigl(I_2^{K^q}\bigr) = \left(\sum_{j=k}^n t_j^r\right)^{1/r} \quad \text{and} \quad d_k\bigl(I_{K_p}^2\bigr) = \left(\sum_{j=k}^n t_j^s\right)^{1/s}.$$

Finally, Theorem 2.1 leads to the following result.

Theorem 3.1. *Let $1 \leq q \leq 2$ and $p \geq 2$ and define positive numbers r, s by $1/r = 1/q - 1/2$ and $1/s = 1/2 - 1/p$. Let $t = (t_1, \ldots, t_n)$ such that $t_1 \geq t_2 \geq \cdots \geq t_n > 0$ and let $K_p = D_t(B_p^n)$ and $K^q = D_t^{-1}(B_q^n)$. Then*

$$\mathrm{r}_i(K^q) = \left(\sum_{j=n-i+1}^n t_j^r\right)^{-1/r} \quad \textit{and} \quad \mathrm{R}_i(K_p) = \left(\sum_{j=n-i+1}^n t_j^s\right)^{1/s}.$$

For $q = 1$ and $p = \infty$, the values of the inner radii of orthogonal cross-polytopes and the outer radii of orthogonal boxes are obtained (see [8, Theorem 4.4]); for $p = q = 2$ the successive radii of the ellipsoids can be deduced (see [18, p. 18]), namely, $\mathrm{R}_i(K_2) = t_{n-i+1}$, $\mathrm{r}_i(K^2) = t_i$.

We also remark that the values of the outer radii of orthogonal cross-polytopes (and so the inner radii of orthogonal boxes) can be derived from [25, Theorem 11.11.7] via Theorem 2.1 (see [8, Proposition 4.3] and [13, Theorem 1]). Finally, we mention that the results from [21, 22] can be used to compute (or to estimate, up to multiplicative constants) the successive radii of unit balls of symmetric n-dimensional normed spaces; in particular this applies to unit balls of Lorentz and Orlicz sequence spaces.

3.1. **Appendix: a geometrical proof of Theorem 1.1.** In this appendix we sketch a geometrical proof of Theorem 1.1. We point out that it partly follows the idea of the proof of [25, Theorem 11.11.4], from a geometric point of view.

In order to show the theorem, we need the following two facts. On the one hand, it is an easy computation to check that

$$\text{If } 1 \leq p \leq 2 \text{ then } \mathrm{R}(B_p^n) = 1. \text{ If } p \geq 2 \text{ then } \mathrm{R}(B_p^n) = n^{1/2-1/p}. \tag{6}$$

On the other hand, we observe that if $P \subset \mathbb{R}^n$ is a polytope with $0 \in \operatorname{int} P$ then, for any $L \in \mathcal{L}_i^n$, $P_L = P \cap L$ is an i-dimensional polytope. Let v be a vertex of P_L and we denote by F the smallest (in the sense of dimension) face of P containing v, which gives $F \cap L = \{v\}$. If we assume that $\dim F > n-i$, then it would be $\dim(F + L) = i + \dim F > n$, which is not possible. Therefore $\dim F \leq n - i$, i.e., we have proved the following:

(7) If $P \subset \mathbb{R}^n$ is a polytope with $0 \in \operatorname{int} P$, then any $L \in \mathcal{L}_i^n$ intersects P in one of its $(n-i)$-faces.

Proof of Theorem 1.1. Notice that in order to prove that $\mathrm{R}_i(B_p^n) = i^{1/2-1/p}$, $p \geq 2$, it suffices to show

$$\mathrm{R}(B_p^n \cap L) \geq i^{1/2-1/p} \quad \text{for all } \; L \in \mathcal{L}_i^n; \tag{8}$$

then, using (6), since $\mathrm{R}(B_p^n|L) \geq \mathrm{R}(B_p^n \cap L)$ and

$$\mathrm{R}\big(B_p^n|\operatorname{lin}\{\mathrm{e}_1, \ldots, \mathrm{e}_i\}\big) = \mathrm{R}\big(B_p^n \cap \operatorname{lin}\{\mathrm{e}_1, \ldots, \mathrm{e}_i\}\big) = i^{1/2-1/p},$$

we get that $\mathrm{R}_i(B_p^n) = i^{1/2-1/p}$, as required.

Let $L \in \mathcal{L}_i^n$. By (7) there exists an $(n-i)$-face F_{n-i} of the cube B_∞^n such that $L \cap F_{n-i} \neq \emptyset$. Let $x \in L \cap F_{n-i}$. Without loss of generality we assume

$$F_{n-i} = \big\{(t_1, \ldots, t_{n-i}, 1, \ldots, 1) \in \mathbb{R}^n : |t_j| \leq 1, j = 1, \ldots, n-i\big\},$$

i.e., $x = (x_1, \ldots, x_{n-i}, 1, \ldots, 1)$ with $|x_j| \leq 1$, $j = 1, \ldots, n-i$. Moreover, let $\lambda = \big(i + \sum_{j=1}^{n-i} |x_j|^p\big)^{-1/p} \in (0, 1]$. Then $z = \lambda x \in L \cap \operatorname{bd} B_p^n$, and since

$p \geq 2$ and $|x_j| \leq 1$, we clearly get

$$|z|_2 = \frac{\left(i + \sum_{j=1}^{n-i} |x_j|^2\right)^{1/2}}{\left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/p}} \geq \frac{\left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/2}}{\left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/p}} = \left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/2-1/p}$$
$$\geq i^{1/2-1/p}.$$

It shows (8). Finally, the value for the inner radii comes from (5) and the fact that $(B_p^n)^* = B_q^n$ with $1/p + 1/q = 1$. □

4. On constant width sets

Constant width sets have been intensively studied along the last century. In the plane they are well known, whereas the situation becomes much more complicated in dimension $n \geq 3$ (see e.g. [5, §15], [12, Ch. 7] and [11] for detailed surveys).

The best known 3-dimensional constant width sets are the revolution of planar convex bodies with constant width, and the so-called Meissner bodies, which are constructed, roughly speaking, in the following way. Let T_3 be a 3-dimensional regular tetrahedron with edge length b, and consider the intersection K of four balls of radius b having the vertices of T_3 as centers. Then K is bounded by four pieces of sphere which meet in six circular arcs. However, K is not a constant width set, because the distance between two of those opposite circular arcs is strictly greater than b. The Meissner bodies are then obtained rounding suitably three of those arcs (see Figure 1). Notice that two Meissner bodies can be constructed, depending on the three rounded arcs either converge to a vertex or form a triangle. For a more detailed construction of the Meissner bodies we refer to [5, p. 144].

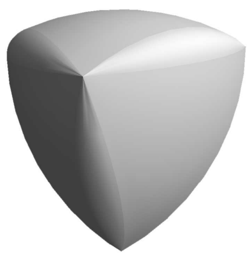

Figure 1. A Meissner body.

Proof of Theorem 1.3. For $K \in \mathcal{W}^n$ with width b, let $L' \in \mathcal{L}_i^n$ be such that

$$\mathrm{r}(K|L'; L') = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L). \tag{9}$$

It is well-known (see e.g. [5, p. 135]) that every orthogonal projection of a constant width set is also a body of constant width having the same width.

Then, using (2) one can easily obtain that

$$\begin{aligned}\mathrm{b} &= \mathrm{R}(K|L') + \mathrm{r}(K|L';L') \geq \mathrm{R}_i(K) + \max_{L\in\mathcal{L}_i^n} \mathrm{r}(K|L;L)\\ &\geq \mathrm{R}_i(K) + \max_{L\in\mathcal{L}_i^n}\max_{x\in L^\perp} \mathrm{r}\big(K\cap(x+L);x+L\big) = \mathrm{R}_i(K)+\mathrm{r}_i(K).\end{aligned}$$

So it remains to prove that the inequality can be strict. Let $K_M \in \mathcal{W}^3$ be a Meissner body with width b. It is known (see e.g. [7, p. 37]) that the orthogonal projection of K_M onto the plane Π determined by two opposite edges of the generating tetrahedron is a 2-dimensional ball with radius b/2. Then, since $\mathrm{R}_2(K_M) \geq \mathrm{R}_1(K_M) = \mathrm{b}/2$ and $\mathrm{R}(K_M|\Pi) = \mathrm{b}/2$, we get $\mathrm{R}_2(K_M) = \mathrm{b}/2$. So, we have to prove that $\mathrm{r}_2(K_M) < \mathrm{b}/2$. In order to show it, we assume $\mathrm{r}_2(K_M) = \mathrm{b}/2$, and we will get a contradiction.

From the definition of $\mathrm{r}_2(K_M)$, there exist $L \in \mathcal{L}_2^3$ and $x \in L^\perp$ such that

$$\frac{\mathrm{b}}{2} = \mathrm{r}_2(K_M) = \mathrm{r}\big(K_M\cap(x+L);x+L\big),$$

and thus there exists a circle C of radius b/2 contained in $K_M \cap (x+L)$. Moreover, observe that $C = K_M \cap (x+L)$, otherwise there would exist a point $p \in \big(K_M\cap(x+L)\big)\backslash C$, and then $\mathrm{D}(K_M) \geq \mathrm{D}\big(K_M\cap(x+L)\big) > \mathrm{b}$, which is not possible. Let $y \in \operatorname{int} K_M$ such that $C = y + (\mathrm{b}/2)B_{i,L}$, and let $v \in \operatorname{relbd} B_{i,L}$. Then the point $y+(\mathrm{b}/2)v \in y + (\mathrm{b}/2)\operatorname{relbd} B_{i,L} = \operatorname{relbd} C$ and thus, $y+(\mathrm{b}/2)v$ cannot be a vertex of K_M.

On the one hand, if $y+(\mathrm{b}/2)v$ lies on one of the four pieces of sphere bounding K_M, by the construction of the Meissner body and taking into account that the segment $\big[y-(\mathrm{b}/2)v, y+(\mathrm{b}/2)v\big] \subset C$ and that it has length b, then $y-(\mathrm{b}/2)v$ should be the opposite vertex, which is not possible. On the other hand, if $y+(\mathrm{b}/2)v$ lies on one of the (rounded) arcs, then C should touch one of the opposite sphere pieces of K_M, which leads to the previous case and again to a contradiction. □

We notice that (9) indeed defines another sequence of successive inner radii, namely,

$$\widetilde{\mathrm{r}}_i(K) = \max_{L\in\mathcal{L}_i^n} \mathrm{r}(K|L;L).$$

See [2] for a detailed study of these and other radii. If these new inner radii are involved, then an equality of the type (2) is obtained. Moreover, it is well-known that for any constant width set $K \in \mathcal{W}^n$ of width b it holds

$$\mathrm{b}\left(1-\sqrt{\frac{n}{2(n+1)}}\right) \leq \mathrm{r}(K) \leq \mathrm{R}(K) \leq \mathrm{b}\sqrt{\frac{n}{2(n+1)}} \tag{10}$$

(see e.g. [11, p. 68] or [12, p. 125]); the analogous result for these inner and the outer radii can be easily obtained.

Proposition 4.1. *For any $K \in \mathcal{W}^n$ of width* b *and all $i = 1,\dots,n$ it holds*

$$\mathrm{R}_i(K) + \widetilde{\mathrm{r}}_i(K) = \mathrm{b}$$

and

$$\mathrm{b}\left(1-\sqrt{\frac{i}{2(i+1)}}\right) < \widetilde{\mathrm{r}}_i(K) \leq \mathrm{R}_i(K) < \mathrm{b}\sqrt{\frac{i}{2(i+1)}}. \tag{11}$$

Proof. Notice that for any $K \in \mathcal{W}^n$, say of width b, and for any $i = 1, \dots, n$, the i-plane $L' \in \mathcal{L}_i^n$ giving the value for $\mathrm{R}_i(K)$ gives also $\widetilde{\mathrm{r}}_i(K)$: indeed, if $\mathrm{R}_i(K) = \mathrm{R}(K|L')$, since $K|L$ is also a constant width set of width b satisfying $\mathrm{R}(K|L) + \mathrm{r}(K|L; L) = \mathrm{b}$ for all $L \in \mathcal{L}_i^n$ (cf. (2)), then

$$\mathrm{r}(K|L'; L') = \mathrm{b} - \mathrm{R}(K|L') \geq \mathrm{b} - \mathrm{R}(K|L) = \mathrm{r}(K|L; L)$$

for all $L \in \mathcal{L}_i^n$, and so $\widetilde{\mathrm{r}}_i(K) = \mathrm{r}(K|L'; L')$. Therefore,

$$\mathrm{R}_i(K) + \widetilde{\mathrm{r}}_i(K) = \mathrm{R}(K|L') + \mathrm{r}(K|L'; L') = \mathrm{b},$$

and moreover, applying (10) to the i-dimensional set $K|L'$ gives the left and right inequalities in (11). In order to conclude the proof of (11) notice that, since K is a constant width set and $\widetilde{\mathrm{r}}_1(K) = \mathrm{D}(K)/2$ (see [2]), then $\widetilde{\mathrm{r}}_i(K) \leq \widetilde{\mathrm{r}}_1(K) = \mathrm{D}(K)/2 = \omega(K)/2 = \mathrm{R}_1(K) \leq \mathrm{R}_i(K)$.

We observe that equality $\widetilde{\mathrm{r}}_i(K) = \mathrm{R}_i(K)$ holds for any constant width set $K \in \mathcal{W}^n$ such that $K|L' = (\mathrm{b}/2)B_{i,L'}$. □

5. A property on p-tangential bodies.

We conclude the paper stating a property for the so-called p-tangential bodies. A convex body $K \in \mathcal{K}^n$ containing the Euclidean ball B_n is called a p-tangential body of B_n, $0 \leq p \leq n-1$, if each support plane of K that is not a support plane of B_n contains only $(p-1)$-singular points of K [30, p. 86]. Here $x \in \operatorname{bd} K$ is said to be an r-singular point of K if the dimension of the normal cone in x is at least $n-r$. For further characterizations and properties of p-tangential bodies we refer to [30, Section 2.2].

So a 0-tangential body of B_n is just B_n itself and each p-tangential body of B_n is also a q-tangential body for $p \leq q \leq n-1$. A 1-tangential body can be seen as the convex hull of B_n and countably many points such that the line segment joining any pair of those points intersects the ball. A celebrated result of Favard [14] states a nice characterization of n-dimensional p-tangential bodies in terms of the so-called quermassintegrals of K, namely, that the $n-p+1$ first ones coincide. We will not enter here in the definition of these measures, for the interested reader we refer to [30, p. 431].

Here we show a result in the spirit of the above mentioned Favard's theorem, in the sense that now, for a p-tangential body, many inner radii also coincide.

Proposition 5.1. *Let $K \in \mathcal{K}^n$ be a p-tangential body of B_n, $0 \leq p \leq n-1$. Then*

$$\mathrm{r}_n(K) = \mathrm{r}_{n-1}(K) = \cdots = \mathrm{r}_{p+1}(K) = 1.$$

Proof. It is a direct consequence from the definition that any p-tangential body of B_n has inradius 1. So, if $p = n-1$ then $\mathrm{r}_n(K) = 1$ and the result follows. Thus, we assume $1 \leq p \leq n-2$.

Since the inner radii form a decreasing sequence then

$$1 = \mathrm{r}_n(K) \leq \mathrm{r}_{n-1}(K) \leq \cdots \leq \mathrm{r}_{p+1}(K),$$

and it suffices to show that $\mathrm{r}_{p+1}(K) \leq 1$. So we assume $\mathrm{r}_{p+1}(K) > 1$ and we will get a contradiction. On the one hand, by definition of inner radii, there exist $t \in \mathbb{R}^n$ and $L \in \mathcal{L}^n_{p+1}$ such that

$$t + \mathrm{r}_{p+1}(K) B_{p+1,L} \subseteq K. \tag{12}$$

On the other hand, in [29, Lemma 2.5] it is shown, in particular, that K is a p-tangential body of B_n, $1 \leq p \leq n-2$, if and only if $K|u^\perp$ is a p-tangential body of $B_{n-1,u^\perp}$ for any unit vector $u \in \mathbb{R}^n$. From this result it can be easily obtained that the orthogonal projection $K|L$ is again a p-tangential body of the ball $B_n|L = B_{p+1,L}$, and then

$$\mathrm{r}(K|L;L) = \mathrm{r}(B_{p+1,L};L) = 1. \tag{13}$$

Moreover, from (12) we get that $t|L + \mathrm{r}_{p+1}(K) B_{p+1,L} \subseteq K|L$, and then, together with (13), we obtain the desired contradiction:

$$1 = \mathrm{r}(K|L;L) \geq \mathrm{r}_{p+1}(K) > 1. \qquad \square$$

This results shows (see [6, Lemma 3.2]) that p-tangential bodies of the Euclidean ball B_n are $\{\mathrm{r}_{p+1}, \ldots, \mathrm{r}_{n-1}\}$-isoradial. We recall that a convex body K is called r_j-isoradial if for every $L \in \mathcal{L}^n_j$ there exist $t \in \mathbb{R}^n$ such that $\big(t + \mathrm{r}_j(K) B_n\big) \cap (t + L) \subset K$, and is said to be $\{\mathrm{r}_j : j \in I\}$-isoradial, for a subset $I \subset \{1, \ldots, n-1\}$, if it is r_j-isoradial for all $j \in I$.

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